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The vortical layer on an inclined cone

By ALBERT G. MUNSON*

Department of Aeronautics and Astronautics, Stanford University, Palo Alto, California

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The problem of flow over a circular cone inclined slightly to a uniform stream is solved using the technique of matched asymptotic expansions. The outer expansion is equivalent to Stone's solution of the problem. The inner expansion, valid in a thin layer near the body, represents Ferri's vortical layer. The solution to first order in angle of attack so obtained is uniformly valid everywhere in the flow field. In the second-order expansion an additional non-uniformity appears near the leeward ray. This defect is removed by inspection. The first-order solution is in agreement with that of Cheng, Woods, Bulakh and Sapunkov. Formulas are given that may be used to render Kopal's numerical result uniformly valid to second order in angle of attack.

1. Introduction

A curious singularity appears at the surface of a circular cone inclined at a small angle to an inviscid supersonic stream. This phenomenon was unknown to Stone (1948, 1952) who expanded the flow quantities formally in ascending powers of the angle of inclination and found the first- and second-order perturbations. His results served as the basis for extensive numerical computation by Kopal (1947*b*, 1949). The singularity was discovered by Ferri (1950) who gave a physical description of the flow near the surface. He deduced that streamlines crossing the shock wave at any circumferential location eventually curve around the body toward the leeward plane of symmetry. Though the entire flow field is rotational, a thin layer of intense vorticity lies near the surface. Ferri called this the vortical layer. All streamlines approach the top ray, and consequently the entropy in that neighbourhood is many-valued. Ferri called this the vortical singularity (see figure 1). He has conjectured that at high angle of attack the singularity leaves the surface of the body. The present flow problem is only one case of the occurrence of vortical singularities. Indeed, they are present in any conical flow without axial symmetry.

Cheng (1962) obtained a mathematical description of Ferri's vortical layer for the circular cone. He first solved Stone's problem using the Newtonian approximation to simplify the analysis, and found a solution correct to first order in angle of attack everywhere except near the body. He then used a modified expansion scheme—not restricted to the Newtonian approximation—to study the flow near the surface. The results thus obtained enabled him to render his formal solution valid to first order in angle of attack even near the body—except

* Now at National Engineering Science Company, Pasadena, California.

possibly in the immediate vicinity of the vortical singularity in the leeward plane of symmetry. Sapunkov (1963*a*) pointed out that Cheng's results for density, shock-wave shape, circumferential velocity, and pressure are incorrect because of an algebraic error.† Bulakh (1962*a*) used Stone's first-order results to study the flow near the surface. His results are similar to, but more general than, those of Cheng. Woods (1962) also used the Stone theory to study the flow near the surface, but did not attempt to give solutions valid in the entire flow field. Sapunkov (1963*b*) used the Newtonian approximation and a modified expansion

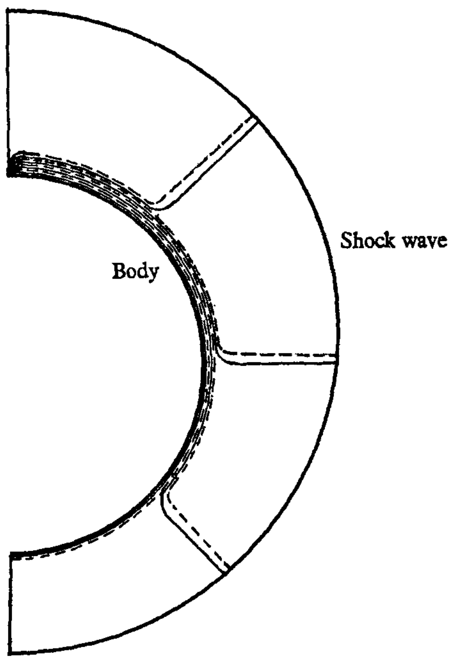

FIGURE 1. Projection of streamlines on r = const. surface (schematic). ——, First-order stream-lines; - - -, second-order streamlines.

scheme to extend his own results (Sapunkov 1963*a*) and those of Cheng to second order in angle of attack. His first-order results differ by higher-order terms—which he subsequently discards—from the results of Cheng (1962) and Sapunkov (1963*a*). He has also corrected a non-uniformity in the gradient of normal velocity component which was not considered by Cheng and Bulakh.

A somewhat different approach to the problem was taken by Gonor (1958), Bulakh (1962*b*), and Sapunkov (1963*b*). Gonor obtained a solution for the outer flow, valid for arbitrary angle of attack using the Newtonian approximation. Bulakh and later Sapunkov treated the non-uniformity at the surface. Their results show the existence of a vortical layer at the surface, but no explicit results for flow quantities are given.

In the present work the problem is studied using the method of matched

† One may obtain the correct result by crossing out terms proportional to $\log(1+k)$ in Cheng's equations (2.7), (2.8), (2.9), (2.10), and (3.19) and adding $2\epsilon(1+k)\log(1+k)$ to (3.18). Also in (2.9) and (2.10) in the same paper, $\log\tau$ should be replaced by -1.

asymptotic expansions (see Lagerstrom 1957; Erdélyi 1961; Van Dyke 1964). Stone's first- and second-order problems—representing the linear and quadratic effects of angle of attack—are shown to provide the first-order and second-order terms in the outer expansion. The non-uniformities in both these terms are studied. A two-term inner expansion is found and matched to the outer expansion. An additional non-uniformity appears in the neighbourhood of the vortical singularity. This is also rendered uniformly valid. The first-order results agree in form with Sapunkov's non-uniformity in the normal velocity component, though they show that he made a computational error. Indeed, it would be impossible to construct a second-order expansion without a knowledge of this non-uniformity. Results otherwise are in agreement with those of Cheng, Bulakh (1962*a*), and Woods. The second-order terms give additional information about the streamline shape, especially in the vicinity of the vortical singularity. Formulas are given that correct Kopal's tabulated results to second order everywhere in the flow field.

Holt (1954) made a local analysis of the flow near the vortical singularity. His results do not agree with either the present work or the results of Cheng, Bulakh, Sapunkov, or Woods.

After the completion of this work the author learned that Mr R. I. Melnik of Grumman Aircraft Company had obtained a solution to this and the related problem of the flow over a cone of elliptic cross-section. The method used is almost identical to that used here, but differs greatly in detail. The results are in agreement.

2. Formulation

2.1. *Co-ordinate system and dimensionless variables*

A spherical co-ordinate system is chosen as shown in figure 2. The angle between the axis and the position vector is ϑ, the distance from the origin is r, and ϕ is the circumferential angle. The velocity at infinity, of magnitude u_∞, lies in the plane $\phi = 0$ and makes an angle α with the axis of the cone. The velocity components u^*, v^*, and w^* are in the directions of increasing r, ϑ, and ϕ, respectively. The pressure and density are p^* and ρ^*. The cone half-angle is τ. Dimensionless variables are defined as

$$u \equiv u^*/u_\infty,\ \ v \equiv v^*/u_\infty \sin\tau,\ \ w \equiv w^*/u_\infty \sin\alpha,\ \ p \equiv p^*/\rho_\infty u_\infty^2 \sin^2\tau,\ \ \rho \equiv \rho^*/\rho_\infty; \tag{2.1}$$

$$\theta \equiv (\sin\vartheta - \sin\tau)/\sin\tau. \tag{2.2}$$

It should be noted that θ is measured from the surface of the cone.

2.2. *Differential equation and boundary conditions*

Substituting the dimensionless variables into the inviscid rotational equations for a perfect gas yields

$$\left[Jv\frac{\partial}{\partial\theta} + \sigma\frac{w}{1+\theta}\frac{\partial}{\partial\phi}\right]u = \sin^2\tau\,(v^2+\sigma w^2), \tag{2.3a}$$

$$\frac{\partial p}{\partial\theta} = \sigma^2\rho\frac{w^2}{1+\theta} - \rho\left[v\frac{\partial}{\partial\theta} + \frac{\sigma w}{J(1+\theta)}\frac{\partial}{\partial\phi} + \frac{u}{J}\right]v, \tag{2.3b}$$

$$\sigma\rho\left[Jv\frac{\partial}{\partial\theta}+\frac{\sigma w}{1+\theta}\frac{\partial}{\partial\phi}+u\right]w=-\frac{1}{1+\theta}\left[\frac{\partial p}{\partial\phi}+\sigma J\rho vw\right], \tag{2.3c}$$

$$2(1+\theta)\rho u+J\frac{\partial}{\partial\theta}[(1+\theta)\rho v]+\sigma\frac{\partial}{\partial\phi}[\rho w]=0, \tag{2.3d}$$

$$\left[Jv\frac{\partial}{\partial\theta}+\sigma\frac{w}{1+\theta}\frac{\partial}{\partial\phi}\right]\frac{p}{\rho^{\gamma}}=0, \tag{2.3e}$$

where $\sigma=\sin\alpha/\sin\tau,\quad J=\cos\vartheta=\{1-\sin^2\tau\,(1+\theta)^2\}^{\frac{1}{2}}.$

In addition, since the flow is isoenergetic, the Bernoulli equation may be used. Thus

$$\tfrac{1}{2}(u^2+v^2\sin^2\tau+\sigma^2w^2\sin^2\tau)+\frac{\gamma}{\gamma-1}\frac{p}{\rho}\sin^2\tau=\frac{1}{2}+\frac{\gamma}{\gamma-1}\frac{p_\infty}{\rho_\infty u_\infty^2}. \tag{2.3f}$$

These equations are not all independent. Indeed, any one except (2.3*d*) can be eliminated. It is convenient, however, in the subsequent analysis, to consider all of them, and at certain points to particularize to a given set of five.

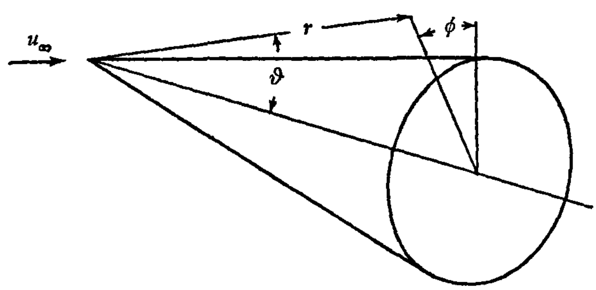

FIGURE 2. The co-ordinate system .

The boundary condition of tangent flow at the surface of the cone is

$$v(0,\phi;\sigma)=0. \tag{2.4}$$

At the bow shock wave the Rankine–Hugoniot relations must be satisfied.

A solution to this problem would furnish a complete flow picture. There are two practical difficulties, however. The differential equations are non-linear and the position of the shock wave is unknown. Because of this latter difficulty, some of the boundary values must be imposed at an unknown surface. In order to circumvent these difficulties, Stone perturbed the well-known basic solution for axisymmetric flow by expanding in powers of the angle of attack.

3. Outer problem

3.1. *The outer solution, expansion in powers of σ*

Stone (1948, 1952) assumed that the dependent variables could be expanded in ascending powers of the angle of attack, the coefficient of each term being itself a Fourier series in the azimuthal angle ϕ. He found that homogeneity of the equations of motion and boundary conditions requires that all Fourier components vanish except those shown below.

$$u(\theta,\phi;\sigma)=u_0(\theta)+\sigma u_{11}(\theta)\cos\phi+\sigma^2(u_{20}(\theta)+u_{22}(\theta)\cos 2\phi)+\ldots, \tag{3.1a}$$

$$v(\theta,\phi;\sigma)=v_0(\theta)+\sigma v_{11}(\theta)\cos\phi+\sigma^2(v_{20}(\theta)+v_{22}(\theta)\cos 2\phi)+\ldots, \tag{3.1b}$$

$$w(\theta,\phi;\sigma) = w_{11}(\theta)\sin\phi+\sigma w_{22}(\theta)\sin 2\phi+\ldots, \tag{3.1c}$$

$$p(\theta,\phi;\sigma) = p_0(\theta)+\sigma p_{11}(\theta)\cos\phi+\sigma^2(p_{20}(\theta)+p_{22}(\theta)\cos 2\phi)+\ldots, \tag{3.1d}$$

$$\rho(\theta,\phi;\sigma) = \rho_0(\theta)+\sigma\rho_{11}(\theta)\cos\phi+\sigma^2(\rho_{20}(\theta)+\rho_{22}(\theta)\cos 2\phi)+\ldots. \tag{3.1e}$$

Substituting (3.1) into (2.3) and equating coefficients of like powers of σ gives the following systems of ordinary differential equations:

for the basic axisymmetric flow

$$u_0' = v_0\sin^2\tau/J, \tag{3.2a}$$

$$p_0'/\rho_0+v_0v_0'+u_0v_0/J = 0, \tag{3.2b}$$

$$v_0'+v_0\{\rho_0'/\rho_0+1/(1+\theta)\}+2u_0/J = 0, \tag{3.2c}$$

$$p_0/\rho_0^\gamma = \text{const.}; \tag{3.2d}$$

for the linear effects of incidence,

$$v_0u_{11}' = v_0\sin^2\tau\, v_{11}/J, \tag{3.3a}$$

$$v_0v_{11}'+v_{11}(u_0v_{11}/J+v_0')+u_{11}v_0/J+p_{11}'/\rho_0-\rho_{11}p_0'/\rho_0^2 = 0, \tag{3.3b}$$

$$v_0w_{11}'+w_{11}\{v_0/(1+\theta)+u_0/J\}-p_{11}/(1+\theta)\rho_0J = 0, \tag{3.3c}$$

$$v_{11}'+v_{11}\left(\frac{\rho_0'}{\rho_0}+\frac{1}{1+\theta}\right)+\frac{2u_{11}}{J}+\frac{w_{11}}{J(1+\theta)}+v_0\left(\frac{\rho_{11}}{p_0}\right)' = 0, \tag{3.3d}$$

$$p_{11}/p_0-\gamma(\rho_{11}/\rho_0) = d, \tag{3.3e}$$

$$u_0u_{11}+v_0v_{11}\sin^2\tau+\frac{p_0}{\rho_0}\frac{\gamma}{\gamma-1}\left[\frac{p_{11}}{p_0}-\frac{\rho_{11}}{\rho_0}\right]\sin^2\tau = 0; \tag{3.3f}$$

and for the quadratic effects of incidence,

$$u_{2m}'-\sin^2\tau\, v_{2m}/J = \{(-1)^{\frac{1}{2}m}/2Jv_0\}w_{11}(u_{11}+w_{11}\sin^2\tau), \tag{3.4a}$$

$$\begin{aligned} v_0v_{2m}'+\frac{p_{2m}'}{\rho_0}+u_0v_{2m}/J+v_0'\left(v_{2m}+\frac{1}{2}\frac{\rho_{11}}{\rho_0}v_{11}+\frac{\rho_{2m}}{\rho_0}v_0\right) \\ +v_{11}'\left(\tfrac{1}{2}v_{11}+\frac{1}{2}\frac{\rho_{11}}{\rho_0}v_0\right)+\frac{v_0}{J}\left(u_{2m}+\frac{1}{2}\frac{\rho_{11}}{\rho_0}u_{11}+\frac{\rho_{2m}}{\rho_0}u_0\right) \\ +\frac{v_{11}}{J}\left(\tfrac{1}{2}u_{11}+\frac{1}{2}\frac{\rho_{11}}{\rho_0}u_0\right)-(-1)^{\frac{1}{2}m}\frac{w_{11}}{(1+\theta)}\left(\frac{v_{11}}{J}+w_{11}\right) = 0, \end{aligned} \tag{3.4b}$$

$$\begin{aligned} v_0w_{22}'+w_{22}\left(\frac{u_0}{J}-\frac{v_0}{1+\theta}\right)+\frac{1}{2}\left(v_{11}w_{11}'+\frac{u_{11}w_{11}}{J}+\frac{w_{11}^2}{J(1+\theta)}\right. \\ \left.+\frac{v_{11}w_{11}}{1+\theta}+\frac{\rho_{11}p_{11}}{J(1+\theta)\rho_0^2}\right)-\frac{2p_{22}}{J(1+\theta)\rho_0} = 0, \end{aligned} \tag{3.4c}$$

$$\begin{aligned} v_{2m}'+v_{2m}\left(\frac{1}{1+\theta}+\frac{\rho_0'}{\rho_0}\right)+\frac{mw_{22}}{J(1+\theta)}+2\frac{u_{2m}}{J}+v_0\left(\frac{\rho_{2m}}{\rho_0}\right)' \\ -\frac{(-1)^{\frac{1}{2}m}}{2}\frac{\rho_{11}w_{11}}{J(1+\theta)\rho_0}+\frac{1}{2}\left(\frac{\rho_{11}}{\rho_0}\right)'\left[v_{11}-v_0\frac{\rho_{11}}{\rho_0}\right] = 0, \end{aligned} \tag{3.4d}$$

$$\frac{p_{2m}}{p_0}-\gamma\frac{\rho_{2m}}{\rho_0}+\frac{\gamma}{4}\left(\frac{\rho_{11}}{\rho_0}\right)^2-\frac{1}{4}\left(\frac{p_{11}}{p_0}\right)^2-(-1)^{\frac{1}{2}m}\int\frac{w_{11}d}{2Jv_0(1+\theta)}d\theta-e_m = 0, \tag{3.4e}$$

$$\begin{aligned} u_0u_{2m}+\tfrac{1}{4}u_{11}^2+(v_0v_{2m}+\tfrac{1}{4}v_{11}^2+\tfrac{1}{4}(-1)^{\frac{1}{2}m}w_{11}^2)\sin^2\tau \\ +\frac{\gamma}{\gamma-1}\frac{p_0}{\rho_0}\left[-\frac{\rho_{2m}}{\rho_0}+\frac{\rho_{11}^2}{\rho_0^2}-\frac{1}{2}\frac{p_{11}}{p_0}\frac{\rho_{11}}{\rho_0}+\frac{1}{2}\frac{p_{2m}}{p_0}\right]\sin^2\tau = 0, \end{aligned} \tag{3.4f}$$

with $m = 0, 2$ in each case.

3.2. *Boundary conditions*

Stone (1948, 1952) obtained boundary conditions at the shock wave by expanding its position in ascending powers of σ about the position for $\sigma = 0$, and transferring the expanded flow variables by Taylor series from the actual shock position to the position for $\sigma = 0$. The present results differ slightly from those given by Stone because he used wind rather than body axes.

The outer expansion will later be seen not to be uniformly valid near the surface, because of the vortical layer. It is therefore not correct to satisfy the boundary condition at the body surface by substituting (3.1*b*) into (2.4). Since, however, no layer exists at zero angle of attack, we can conclude that

$$v_0(0) = 0. \tag{3.5a}$$

Woods pointed out that if wind axes are used one encounters difficulty in formulating the surface boundary condition because of the non-uniformity in the expansion for the normal velocity component. This is avoided here by the use of body axes.

It will be shown later that the correct boundary conditions for the higher-order perturbations are

$$v_{11}(0) = 0, \tag{3.5b}$$

$$v_{2m}(0) = 0, \quad m = 0, 2. \tag{3.5c}$$

These conditions would be obtained if one substituted (3.1*b*) into (2.4).

3.3. *Behaviour of the outer expansion near the surface*

The non-linear equations (3.2) governing the basic axisymmetric flow were solved numerically by Taylor & Maccoll (1933) and Kopal (1947*a*). The equations governing the linear (3.3) and second-order (3.4) effects have been solved numerically by Kopal (1947*b*, 1949). Roberts & Riley (1954) have given formulas that can be used to transform the numerical results of Kopal to the present co-ordinate system. Thus solutions to the systems (3.2), (3.3) and (3.4) are available in the literature.

We will now examine these systems of equations and ascertain the behaviour of the solutions near the surface. From (3.5*a*) and (3.2*c*) it follows that for small θ, $v_0 \to 0$ while v_0' is finite. Therefore near the surface we have

$$v_0' = \theta v_0'(0) + \ldots. \tag{3.6}$$

From (3.6) both the first- and second-order equations possess coefficients that vanish near the surface. The possibility therefore arises that some of the first- or second-order quantities are singular near the surface. Since both of these systems are linear and the individual equations are first order, they may be integrated formally and the resultant integrals examined near the surface. Stone (1948) has shown that the first-order quantities are regular at the surface. Clearly, from (3.4*e*) and (3.6) the second-order correction to either the pressure or density or both is singular at the surface. Indeed, for small θ we have

$$\frac{p_{2m}}{p_0} - \gamma\frac{\rho_{2m}}{\rho_0} = (-1)^{\frac{1}{2}m}\int\frac{w_{11}d}{2Jv_0}\,d\theta + \text{terms finite at surface.} \tag{3.7}$$

It can be shown that ρ_{2m} and u_{2m} are the only second-order quantities that are singular at the surface ($dv_{2m}/d\theta$ is singular, however). So we have

$$\frac{\rho_{2m}}{\rho_0} = -(-1)^{\frac{1}{2}m} \frac{w_{11}(0)\, d}{2\gamma J(0)\, v_0'(0)} \log\theta + \text{terms finite at surface,} \tag{3.8a}$$

$$u_{2m} = (-1)^{\frac{1}{2}m} \frac{w_{11}(0)\, [u_{11}(0) + w_{11}(0) \sin^2\tau]}{2J(0)\, v_0'(0)} \log\theta + \text{terms finite at surface.} \tag{3.8b}$$

Some useful relations between the zero-incidence, first-order, and second-order quantities at the surface are

$$v_0' = -2u_0/J, \tag{3.9}$$

$$u_0 w_{11} = p_{11}/\rho_0, \tag{3.10a}$$

$$v_{11}' = -2u_{11}/J - w_{11}/J, \tag{3.10b}$$

$$\frac{\rho_0 u_0}{p_0 \sin^2\tau} \frac{\gamma-1}{\gamma} u_{11} = \frac{\rho_{11}}{\rho_0} - \frac{p_{11}}{p_0}, \tag{3.10c}$$

$$u_0 w_{22} + \tfrac{1}{2}(u_{11} w_{11} + w_{11}^2 + \rho_{11} p_{11}/\rho_0^2) - 2p_{22}/\rho_0 = 0, \tag{3.11a}$$

$$v_{2m}' + m w_{22}/J + 2u_{2m}/J + v_0(\rho_{2m}/\rho_0)' - (-1)^{\frac{1}{2}m} \rho_{11} w_{11}/2J\rho_0 = 0, \tag{3.11b}$$

$$u_0 u_{2m} + \tfrac{1}{4}u_{11}^2 + \tfrac{1}{4}(-1)^{\frac{1}{2}m} w_{11}^2 \sin^2\tau + \frac{\gamma}{\gamma-1} \frac{p_0}{\rho_0} \left[-\frac{\rho_{2m}}{\rho_0} + \frac{1}{2} \frac{\rho_{11}^2}{\rho_0^2} - \frac{1}{2} \frac{p_{11}}{p_0} \frac{\rho_{11}}{\rho_0} + \frac{p_{2m}}{p_0} \right] \sin^2\tau = 0, \tag{3.11c}$$

where $J(0) = \cos\tau$. Stone (1952) and later Cheng (1962) noted this singular behaviour of the second-order quantities but Kopal (1949) ignored it when he obtained numerical solutions to the system. This behaviour is an indication that some of the quantities are not given correctly at the surface to second order by the outer expansion. From the following expression for entropy it will be seen that even the first-order variables are not given correctly there. If we define Δs to be the difference between the value of entropy at incidence and the value at zero incidence, then

$$\frac{\Delta s}{c_v} = \log \left[\frac{p}{p_0} \left(\frac{\rho_0}{\rho} \right)^\gamma \right]. \tag{3.12}$$

Substituting from (3.1) gives

$$\frac{\Delta s}{c_v} = \sigma d \cos\phi + \sigma^2 \left\{ \frac{p_{20}}{p_0} - \gamma \frac{\rho_{20}}{\rho_0} + \frac{\gamma}{4} \left(\frac{\rho_{11}}{\rho_0} \right)^2 - \frac{1}{4} \left(\frac{p_{11}}{p_0} \right)^2 + \left[\frac{p_{22}}{p_0} - \gamma \frac{\rho_{22}}{\rho_0} - \frac{\gamma}{4} \left(\frac{\rho_{11}}{\rho_0} \right)^2 + \frac{1}{4} \left(\frac{p_{11}}{p_0} \right)^2 \right] \cos 2\phi \right\} + \ldots. \tag{3.13}$$

It can be concluded directly that (3.13) predicts an entropy variation on the body surface to first order, whereas we know the entropy should be constant. From (3.3e), the coefficient of $\sigma \cos\phi$ in (3.13) does not vary with θ and hence can be computed from the shock-wave relations. From these relations it follows that only for shock waves of zero strength is this coefficient zero. The second-order correction to the entropy is of course infinite at the surface.

3.4. *Region of non-uniformity*

The form of the singularity, i.e. the fact that $\rho_{2m} \sim \log\theta$, suggests that near the surface the first-order density perturbation is proportional to θ^σ. This in turn suggests that the behaviour of the solution near the body can be described better in terms of θ^σ than in terms of θ itself.

This same conclusion can be arrived at by other considerations. In reducing (2.3*e*) to (3.3*e*) it is necessary to drop

$$\sigma\frac{w_{11}}{1+\theta}\sin^2\phi\left(\frac{p_{11}}{p_0}-\gamma\frac{\rho_{11}}{\rho_0}\right)$$

while retaining

$$\frac{v_0\cos\phi}{J}\frac{\partial}{\partial\theta}\left(\frac{p_{11}}{p_0}-\gamma\frac{\rho_{11}}{\rho_0}\right).$$

As long as θ does not approach zero the former is smaller than the latter, but near the surface the situation may be reversed since by (3.6) v_0 is proportional to θ for small θ. We must therefore consider a small region in which $\theta\,(\partial/\partial\theta) = O(\sigma)$. If we define $\Theta \equiv \theta^\sigma$, we magnify the region such that $\Theta(\partial/\partial\Theta) = O(1)$.

4. Inner problem

4.1. *Inner variables and differential equations in inner variables*

New independent variables that are of order unity in the vortical layer are defined according to

$$\Theta = \theta^\sigma, \tag{4.1a}$$

$$\Phi = \phi. \tag{4.1b}$$

To exhibit the zero in the velocity we define

$$V = v/\theta. \tag{4.2a}$$

This makes V of $O(1)$ in the region of interest. The other dependent variables are defined according to

$$U = u, \tag{4.2b}$$

$$W = w, \tag{4.2c}$$

$$P = p, \tag{4.2d}$$

$$R = \rho. \tag{4.2e}$$

We substitute the above into (2.3) and obtain

$$\left[\sigma J V\Theta\frac{\partial}{\partial\Theta}+\sigma\frac{W}{1+\Theta^{1/\sigma}}\frac{\partial}{\partial\Phi}\right]U = \sin^2\tau\,[\Theta^{2/\sigma}V^2+\sigma^2W^2], \tag{4.3a}$$

$$\sigma\Theta\frac{\partial P}{\partial\Theta} = \Theta^{1/\sigma}\left\{\sigma^2R\frac{W^2}{1+\Theta^{1/\sigma}}-R\left[\sigma V\Theta\frac{\partial}{\partial\Theta}+\sigma\frac{W}{J(1+\Theta^{1/\sigma})}\frac{\partial}{\partial\Phi}+\frac{U}{J}\right](\Theta^{1/\sigma}V)\right\}, \tag{4.3b}$$

$$\sigma R\left[\sigma J V\Theta\frac{\partial}{\partial\Theta}+\sigma\frac{W}{1+\Theta^{1/\sigma}}\frac{\partial}{\partial\Phi}+U\right]W = \frac{1}{1+\Theta^{1/\sigma}}\left[\frac{\partial P}{\partial\Phi}+\sigma\Theta^{1/\sigma}JRVW\right], \tag{4.3c}$$

$$2(1+\Theta^{1/\sigma})RU+J(1+\Theta^{1/\sigma})VR+J\sigma\Theta\,\partial[(1+\Theta^{1/\sigma})VR]/\partial\Theta+\sigma\,\partial(RW)/\partial\Phi = 0, \tag{4.3d}$$

$$\left[\sigma J V\Theta\frac{\partial}{\partial\Theta}+\sigma\frac{W}{1+\Theta^{1/\sigma}}\frac{\partial}{\partial\Phi}\right]\frac{P}{R^\gamma} = 0, \tag{4.3e}$$

$$\tfrac{1}{2}(u^2+\Theta^{2/\sigma}V^2\sin^2\tau+\sigma^2W^2\sin^2\tau)+\frac{\gamma}{\gamma-1}\frac{P}{R}\sin^2\tau = \frac{1}{2}+\frac{\gamma}{\gamma-1}\frac{p_\infty}{\rho_\infty u_\infty^2}. \tag{4.3f}$$

4.2. *Inner expansions*

We expand the inner variables in ascending powers of σ according to

$$W(\Theta, \Phi; \sigma) = \sum_{n=1}^{\infty} \sigma^{n-1} W_n(\Theta, \Phi), \tag{4.4a}$$

$$U(\Theta, \Phi; \sigma) = \sum_{n=0}^{\infty} \sigma^{n} U_n(\Theta, \Phi), \tag{4.4b}$$

$$V(\Theta, \Phi; \sigma) = \sum_{n=0}^{\infty} \sigma^{n} V_n(\Theta, \Phi), \tag{4.4c}$$

$$P(\Theta, \Phi; \sigma) = \sum_{n=0}^{\infty} \sigma^{n} P_n(\Theta, \Phi), \tag{4.4d}$$

$$R(\Theta, \Phi; \sigma) = \sum_{n=0}^{\infty} \sigma^{n} R_n(\Theta, \Phi). \tag{4.4e}$$

We substitute these expressions into equations (4.3) and equate coefficients of like powers of σ. The zero-incidence equations simply express the fact that these quantities are not functions of Θ, i.e. they do not change across the layer. This is the result we expect intuitively. We find the following systems of equations for the linear effects of incidence

$$[JV_0\Theta(\partial/\partial\Theta) + W_1(\partial/\partial\Phi)]\, U_1 = W_1^2 \sin^2\tau, \tag{4.5a}$$

$$\partial P_1/\partial\Theta = 0, \tag{4.5b}$$

$$R_0 U_0 W_1 = -\partial P_1/\partial\Phi, \tag{4.5c}$$

$$2U_1 + JV_1 + (\partial W_1/\partial\Phi) = 0, \tag{4.5d}$$

$$\left[JV_0\Theta\frac{\partial}{\partial\Theta} + W_1\frac{\partial}{\partial\Phi}\right]\left(\frac{P_1}{P_0} - \gamma\frac{R_1}{R_0}\right) = 0, \tag{4.5e}$$

$$U_0U_1 + \frac{\gamma}{\gamma-1}\frac{P_0}{R_0}\left[\frac{P_1}{P_0} - \frac{R_1}{R_0}\right]\sin^2\tau = 0; \tag{4.5f}$$

and for the quadratic effects of incidence

$$[JV_0\Theta(\partial/\partial\Theta) + W_1(\partial/\partial\Phi)]\, U_2 + [JV_1\Theta(\partial/\partial\Theta) + W_2(\partial/\partial\Phi)]\, U_1 = 2\sin^2\tau W_1 W_2, \tag{4.6a}$$

$$\partial P_2/\partial\Theta = 0, \tag{4.6b}$$

$$W_2 = -\frac{1}{R_0U_0}\frac{\partial}{\partial\Phi}P_2 - W_1\frac{U_1}{U_0} - \frac{R_1}{R_0}W_1 - \frac{W_1}{U_0}\frac{\partial}{\partial\Phi}W_1, \tag{4.6c}$$

$$V_2 = -\frac{2U_2}{J} - \frac{1}{J\gamma}W_1\frac{\partial}{\partial\Phi}\frac{P_1}{P_0} - \Theta\frac{\partial}{\partial\Theta}V_1 - \frac{1}{J}\frac{\partial}{\partial\Phi}W_2, \tag{4.6d}$$

$$\left[JV_0\Theta\frac{\partial}{\partial\Theta} + W_1\frac{\partial}{\partial\Phi}\right]\left[\frac{P_2}{P_0} - \gamma\frac{R_2}{R_0} + \frac{\gamma}{2}\left(\frac{R_1}{R_0}\right)^2 - \frac{1}{2}\left(\frac{P_1}{P_0}\right)^2\right] + \left[JV_1\Theta\frac{\partial}{\partial\Theta} + W_2\frac{\partial}{\partial\Phi}\right]\left[\frac{P_1}{P_0} - \gamma\frac{R_1}{R_0}\right] = 0, \tag{4.6e}$$

$$U_0U_2 + \tfrac{1}{2}U_1^2 + \tfrac{1}{2}W_1^2\sin^2\tau + \frac{P_0}{R_0}\frac{\gamma}{\gamma-1}\left[-\frac{R_2}{R_0} + \frac{R_1^2}{R_0^2} - \frac{P_1}{P_0}\frac{R_1}{R_0} + \frac{P_2}{P_0}\right]\sin^2\tau = 0. \tag{4.6f}$$

Note that the terms of $O(\Theta^{1/\sigma})$ do not contribute to the inner solution. This is consistent with neglecting exponential terms in classical boundary-layer theory.

4.3. *Boundary and matching condition*

The boundary condition at the surface is obtained by requiring that (2.4) be satisfied. This is shown in detail in § 5.3.

The inner expansion is valid only in a thin layer near the body and cannot be expected to satisfy the shock conditions. Therefore the boundary conditions are obtained by matching with the outer expansion. We apply the following matching principle:

$$\begin{aligned}&m\text{-term inner expansion of } (p\text{-term outer expansion})\\ &\qquad = p\text{-term outer expansion of } (m\text{-term inner expansion}).\end{aligned} \tag{4.7}$$

As an example of the application of the matching principle consider the three-term outer and inner expansions for the pressure.

$$\begin{aligned}&3\text{-term inner expansion of } [p_0(\theta)+\sigma p_{11}(\theta)\cos\phi+\sigma^2(p_{20}+p_{22}\cos 2\phi)]\\ &\qquad = 3\text{-term outer expansion of } [P_0(\Theta,\Phi)+\sigma P_1(\Theta,\Phi)+\sigma^2 P_2(\Theta,\Phi)].\end{aligned} \tag{4.8}$$

We construct the outer expansion of the first bracket by rewriting the terms therein in the variables Θ and Φ (e.g. $p_{11}(\Theta^{1/\sigma})\cos\Phi$) and expanding in powers of σ. Therefore we obtain

$$\begin{aligned}&p_0(0)+\sigma p_{11}(0)\cos\Phi+\sigma^2\{p_{20}(0)+p_{22}(0)\cos 2\Phi\}\\ &\qquad = 3\text{-term outer expansion of } [P_0(\Theta,\Phi)+\sigma P_1(\Theta,\Phi)+\sigma^2 P_2(\Theta,\Phi)].\end{aligned} \tag{4.9}$$

Similarly, we take the outer expansion of the expression in brackets in (4.9) by writing the functions in inner variables and expanding in powers of σ. See § 5.3 for examples of this.

5. Uniformly valid solutions

5.1. *General procedure*

In what follows we will solve the system of equations governing the first-order perturbations, apply the matching and boundary conditions and immediately modify each inner expansion so that the result is uniformly valid throughout the flow field. We will solve the second-order problem in the same fashion. Before we do this, however, we will first obtain a solution for the pressure.

5.2. *Pressure*

Pressure is in a class by itself and can be disposed of first. The equations for the first- and second-order pressure perturbations state that these quantities are independent of Θ. From this fact and (4.9) we obtain

$$P_0 = p_0(0), \tag{5.1a}$$

$$P_1 = p_{11}(0)\cos\Phi, \tag{5.1b}$$

$$P_2 = p_{20}(0)+p_{22}(0)\cos 2\Phi. \tag{5.1c}$$

Then the outer expansion (3.1*d*) is uniformly valid to second order everywhere. This result is not restricted to second order. From (4.3*b*) it follows that

$$\partial P_n/\partial\Theta = 0 \quad \text{for all } n. \tag{5.2}$$

Therefore the outer expansion for pressure is uniformly valid to any order. This is in contrast to viscous boundary-layer theory where the second-order correction to the pressure varies across the layer (Van Dyke 1962).

5.3. *First-order solution*

We shall now solve (4.5) for the first-order perturbations. Since P is known from (5.1*b*), equations (4.5*a*), (4.5*c*), (4.5*d*) and (4.5*f*) are sufficient to determine the unknowns. First, we evaluate the basic axisymmetric flow quantities. Because these quantities do not change across the layer the outer expansions are valid at the surface. Therefore we can write

$$U_0 = u_0(0), \tag{5.3a}$$

$$R_0 = \rho_0(0), \tag{5.3b}$$

$$V_0 = v_0'(0). \tag{5.3c}$$

We have from (4.5*c*)

$$W_1 = p_{11}(0)\sin\Phi/\rho_0(0)\,u_0(0), \tag{5.4a}$$

and from (3.10*a*)

$$W_1 = w_{11}(0)\sin\Phi. \tag{5.4b}$$

Since the coefficients of (4.5*a*) are now known we can integrate this equation, using the theory of characteristics for linear partial differential equations of first order. The result is

$$u_1 = f(\zeta_1) - w_{11}\sin^2\tau\cos\Phi, \tag{5.5a}$$

where f is an arbitrary function and

$$\zeta_1^2 = \Theta^{w_{11}(0)\sec\tau/v_0'(0)}\,[(1+\cos\Phi)/(1-\cos\Phi)]. \tag{5.5b}$$

We must evaluate f from the matching condition (4.7). This condition requires that

$$u_0(0) + \sigma u_{11}(0)\cos\Phi = \text{2-term outer expansion of } (U_0 + \sigma U_1). \tag{5.6}$$

We now consider the function $(1-\zeta_1^2)/(1+\zeta_1^2)$. We write this function in terms of θ and ϕ, expand in ascending powers of σ, and retain the first term. We obtain

$$\text{1-term outer expansion of } [(1-\zeta_1^2)/(1+\zeta_1^2)] = -\cos\Phi.$$

Therefore to satisfy (5.6) we must choose f as

$$f(\zeta_1) = -(u_{11}(0) + w_{11}(0)\sin^2\tau)(1-\zeta_1^2)/(1+\zeta_1^2). \tag{5.7}$$

From (4.5*f*) (and substituting from (3.10)) we find that†

$$\frac{R_1}{R_0} = \frac{d}{\gamma}\frac{1-\zeta_1^2}{1+\zeta_1^2} + \frac{p_{11}(0)}{\gamma p_0(0)}\cos\Phi; \tag{5.8}$$

also from (4.5*d*) we find

$$V_1 = [2/J(0)]\,[u_{11}(0) + w_{11}(0)\sin^2\tau]\,(1-\zeta_1^2)/(1+\zeta_1^2) + [w_{11}(0)/J(0)]\cos\Phi\,(2\sin^2\tau - 1). \tag{5.9}$$

It follows, from a comparison of (5.9), (3.10*b*) and the definition of V (4.2*a*), that V_1 matches the first perturbation term in the outer expansion. It also follows, from the fact that u_{11} and w_{11} are well behaved at the surface, that the normal component of the velocity, θV_1, vanishes at the surface. Thus the surface-boundary condition is satisfied.

† Many alternative forms of this and the following equations can be obtained using (3.9), (3.10) and (3.11).

Uniformly valid expansions for the density and radial velocity are given by

$$u = u_0 - \sigma\{[u_{11}(\theta) + w_{11}(\theta)\sin^2\tau][(1-\zeta_1^2)/(1+\zeta_1^2)] + w_{11}(\theta)\cos\Phi\sin^2\tau\} + \ldots, \quad (5.10)$$

$$\rho = \rho_0 + \sigma\rho_0\left\{\frac{d}{\gamma}\frac{1-\zeta_1^2}{1+\zeta_1^2} + \frac{p_{11}(\theta)}{\gamma p_0(\theta)}\cos\Phi\right\} + \ldots. \quad (5.11)$$

To construct a uniformly valid expansion for the normal velocity we use the following rule for each term

uniformly valid expansion = inner expansion + outer expansion
− inner expansion of (outer expansion).

Therefore we obtain

$$v = v_0(\theta) + \sigma\theta\{2[u_{11}(0) + w_{11}(0)\sin^2\tau]\sec\tau\,(1-\zeta_1^2)/(1+\zeta_1^2)$$
$$+ w_{11}(0)\cos\Phi\,(2\sin^2\tau - 1)\sec\tau - v_{11}'(0)\cos\Phi + (1/\theta)\,v_{11}(\theta)\cos\Phi\} + \ldots. \quad (5.12)$$

To first order in σ the outer expansions (3.1 c) and (3.1 d) for w and p are uniformly valid. The uniformly valid first-order expansion for the entropy can be found by substituting (5.11) and the first two terms of (3.1 d) into equation (3.12). The result is

$$\Delta s/c_v = -\sigma d(1-\zeta_1^2)/(1+\zeta_1^2) + \ldots. \quad (5.13)$$

Equation (5.13) agrees with (3.13) in the region where the latter is valid and predicts a constant value of entropy on the body surface. First-order streamlines are lines of constant ζ_1 (see figure 1).

5.4. *Second-order solution*

The system (4.6) excepting (4.6 e) is solved in the same manner as was (4.5). That is, W_2 is evaluated from (4.6 c) and (5.1 c). Equation (4.6 a) is then integrated and the matching condition (4.7) applied. The resulting second-order expansion is

$$u = u_0(\theta) - \sigma\left\{[u_{11}(\theta) + w_{11}(\theta)\sin^2\tau]\frac{1-\zeta_1^2}{1+\zeta_1^2} + w_{11}(\theta)\cos\Phi\sin^2\tau\right\}$$
$$+\sigma^2\left\{2\left[u_{22}(\theta) + \frac{1}{2}\frac{w_{11}(\theta)}{v_0'(\theta)}(u_{11}(\theta) + w_{11}(\theta)\sin^2\tau)\sec\tau\log\theta\right]\left(\frac{1-\zeta_1^2}{1+\zeta_1^2}\right)^2\right.$$
$$+ u_{20}(\theta) - u_{22}(\theta) - \frac{w_{11}(\theta)}{v_0'(\theta)}(u_{11}(\theta) + w_{11}(\theta)\sin^2\tau)\sec\tau\log\theta$$
$$+\frac{\sin^2\tau}{2}\left[\frac{4p_{22}(\theta)}{\rho_0(\theta)\,u_0(\theta)} + \frac{w_{11}^2(\theta)}{u_0(\theta)}\sin^2\tau + \frac{w_{11}(\theta)}{\gamma}\frac{p_{11}(\theta)}{p_0(\theta)} + \frac{w_{11}^2(\theta)}{u_0(\theta)}\right]$$
$$\times\left[\left(\frac{1-\zeta_1^2}{1+\zeta_1^2}\right)^2 - \cos^2\Phi\right] - \sin^2\tau\left[\frac{w_{11}(\theta)\,u_{11}(\theta)}{u_0(\theta)} + \frac{w_{11}^2(\theta)}{u_0(\theta)}\sin^2\tau - \frac{w_{11}(\theta)\,d}{\gamma}\right]$$
$$\times\left[\cos\Phi\frac{1-\zeta_1^2}{1+\zeta_1^2} + \left(\frac{1-\zeta_1^2}{1+\zeta_1^2}\right)^2\right] + \frac{\sec\tau}{v_0'(\theta)}(u_{11}(\theta) + w_{11}(\theta)\sin^2\tau)$$
$$\times\left[2\frac{w_{11}(\theta)\,u_{11}(\theta)}{u_0(\theta)} + 2\frac{w_{11}^2(\theta)}{u_0(\theta)}\sin^2\tau - \frac{w_{11}(\theta)\,d}{\gamma}\right]\left[\frac{4\zeta_1^2}{(1+\zeta_1^2)^2}\frac{1-\zeta_1^2}{1+\zeta_1^2}\log\Theta\right]$$
$$+\frac{1}{2w_{11}(\theta)}(u_{11}(\theta) + w_{11}(\theta)\sin^2\tau)\left[\frac{4p_{22}(\theta)}{\rho_0(\theta)\,u_0(\theta)} - 2\frac{w_{11}^2(\theta)}{u_0(\theta)}\sin^2\tau\right.$$
$$\left.\left.-\frac{w_{11}(\theta)}{\gamma}\frac{p_{11}(\theta)}{p_0(\theta)} - \frac{3}{2}\frac{w_{11}^2(\theta)}{u_0(\theta)}\right]\left[\frac{4\zeta_1^2}{(1+\zeta_1^2)^2}\right]\left[\log\sin^2\Phi - \log\frac{4\zeta_1^2}{(1+\zeta_1^2)^2}\right]\right\} + \ldots. \quad (5.14)$$

V_2 and R_2 can now be evaluated from (4.6d) and (4.6f). Before this is done it is desirable to examine (5.14). It can be seen that u_{22} and u_{20} always occur in combination with terms proportional to $\log\theta$. These terms arise as a result of the application of the matching principle. Equation (3.8b) shows that u_{22} and u_{20} are themselves proportional to $\log\theta$ near the surface and a comparison of this equation with (5.14) shows that the terms in question are well behaved near the surface. All other functions in (5.14) are well behaved near the surface except the terms proportional to $\log\sin\Phi$ and $\log\Theta$. Because of the functions of ζ_1 that multiply these two logarithmic terms, they can be singular only at $\theta = 0$, $\phi = 0$; that is, near the leeward ray. Thus we have reduced the region of non-uniformity from the entire cone surface to the neighbourhood of the top ray.

This additional non-uniformity can be treated by observing that, as a result of the expansion procedure, a term of the form Θ^σ would appear as

$$1+\sigma\log\Theta+\dots.$$

First, we notice the following:

$$\log\sin^2\Phi = -\log\zeta_1^2+\log\Theta^{2w_{11}\sec\tau/v_0'}+\log(1+\cos\Phi)^2, \tag{5.15a}$$

$$\frac{1-\{(1+\cos\Phi)/(1-\cos\Phi)\}\,\Theta^{f_1(\theta)+\sigma f_2(\theta)}}{1+\{(1+\cos\Phi)/(1-\cos\Phi)\}\,\Theta^{f_1(\theta)+\sigma f_2(\theta)}} = \frac{1-\{(1+\cos\Phi)/(1-\cos\Phi)\}\,\Theta^{f_1(\theta)}}{1+\{(1+\cos\Phi)/(1-\cos\Phi)\}\,\Theta^{f_1(\theta)}}$$
$$-2f_2(\theta)\,\sigma\frac{\{(1+\cos\Phi)/(1-\cos\Phi)\}\,\Theta^{f_1(\theta)}}{1+\{(1+\cos\Phi)/(1-\cos\Phi)\}\,\Theta^{f_1(\theta)}}\log\Theta+\dots, \tag{5.15b}$$

$$1-\Theta^{\sigma f_3(\theta)} = -\sigma f_3(\theta)\log\Theta+\dots. \tag{5.15c}$$

Equation (5.14) can now be rewritten, using the above relations, in the form

$$\begin{aligned}
u = u_0(\theta)-\sigma\Bigg\{&[u_{11}(\theta)+w_{11}(\theta)\sin^2\tau]\frac{1-\zeta_2^2}{1+\zeta_2^2}+w_{11}(\theta)\sin^2\tau\cos\Phi\\
&+\frac{v_0'(\theta)}{4w_{11}^2(\theta)\sec\tau}(u_{11}(\theta)+w_{11}(\theta)\sin^2\tau)\left[2\frac{w_{11}(\theta)\,u_{11}(\theta)}{u_0(\theta)}+2\frac{w_{11}^2(\theta)}{u_0(\theta)}\sin^2\tau\right.\\
&\left.-\frac{w_{11}(\theta)\,d}{\gamma}\right](1-\Theta^{4\sigma w_{11}^2(\theta)\sec\tau/[v_0'(\theta)]^2})\frac{4\zeta_2^2}{(1+\zeta_2^2)^2}\frac{1-\zeta_2^2}{1+\zeta_2^2}\Bigg\}\\
&+\sigma^2\Bigg\{\left[2u_{22}(\theta)+\frac{w_{11}(\theta)}{v_0'(\theta)}(u_{11}(\theta)+w_{11}(\theta)\sin^2\tau)\sec\tau\log\theta\right]\left(\frac{1-\zeta_2^2}{1+\zeta_2^2}\right)^2\\
&+u_{20}(\theta)-u_{22}(\theta)-\frac{w_{11}(\theta)}{v_0'(\theta)}(u_{11}(\theta)+w_{11}(\theta)\sin^2\tau)\sec\tau\log\theta\\
&+\frac{\sin^2\tau}{2}\left[\frac{4p_{22}(\theta)}{\rho_0(\theta)\,u_0(\theta)}+\frac{w_{11}^2(\theta)}{u_0(\theta)}\sin^2\tau+\frac{w_{11}(\theta)}{\gamma}\frac{p_{11}(\theta)}{p_0(\theta)}+\frac{w_{11}^2(\theta)}{u_0(\theta)}\right]\left[\left(\frac{1-\zeta_2^2}{1+\zeta_2^2}\right)^2-\cos^2\Phi\right]\\
&-\sin^2\tau\left[\frac{w_{11}(\theta)\,u_{11}(\theta)}{u_0(\theta)}+\frac{w_{11}^2(\theta)}{u_0(\theta)}\sin^2\tau-\frac{w_{11}(\theta)\,d}{\gamma}\right]\left[\cos\Phi\frac{1-\zeta_2^2}{1+\zeta_2^2}+\left(\frac{1-\zeta_2^2}{1+\zeta_2^2}\right)^2\right]\\
&+\frac{1}{2w_{11}(\theta)}(u_{11}(\theta)+w_{11}(\theta)\sin^2\tau)\left[\frac{4p_{22}(\theta)}{\rho_0(\theta)\,u_0(\theta)}-2\frac{w_{11}^2(\theta)}{u_0(\theta)}\sin^2\tau\right.\\
&\left.-\frac{w_{11}(\theta)}{\gamma}\frac{p_{11}(\theta)}{p_0(\theta)}-\frac{3}{2}\frac{w_{11}^2(\theta)}{u_0(\theta)}\right]\frac{4\zeta_2^2}{(1+\zeta_2^2)^2}\left[\log(1+\cos\Phi)^2-\log\frac{4\zeta_2^2}{(1+\zeta_2^2)^2}\right]\Bigg\}+\dots,
\end{aligned} \tag{5.16a}$$

$$\zeta_2^2 = \left(\frac{1+\cos\Phi}{1-\cos\Phi}\right)\Theta^A, \quad \text{where} \quad A = \frac{2w_{11}(\theta)\sec\tau}{v_0'(\theta)}\left[1 - \frac{\sigma}{w_{11}(\theta)}\left(\frac{4p_{22}(\theta)}{\rho_0(\theta)\,u_0(\theta)}\right.\right.$$
$$\left.\left. + 2\frac{w_{11}^2(\theta)}{u_0(\theta)}\sin^2\tau - \frac{w_{11}(\theta)}{\gamma}\frac{p_{11}(\theta)}{p_0(\theta)} - \frac{3}{2}\frac{w_{11}^2(\theta)}{u_0(\theta)}\right)\right]. \quad (5.16b)$$

The equation above is uniformly valid even in the neighbourhood of the vortical singularity. Elsewhere it agrees with (5.14) to second order in σ.

The previously determined uniformly valid second-order expansion for the circumferential velocity is

$$w = w_{11}(\theta)\sin\Phi + \sigma\left\{\frac{1}{2}\left(2w_{22}(\theta) + \frac{u_{11}(\theta)\,w_{11}(\theta)}{u_0(\theta)}\right.\right.$$
$$\left. + \frac{p_{11}(\theta)\,\rho_{11}(\theta)}{\rho_0^2(\theta)\,u_0(\theta)} + \frac{w_{11}^2(\theta)}{u_0(\theta)}\sin^2\tau - \frac{1}{\gamma}w_{11}(\theta)\frac{p_{11}(\theta)}{p_0(\theta)}\right)\sin 2\Phi$$
$$\left. + \left(\frac{w_{11}(\theta)\,u_{11}(\theta)}{u_0(\theta)} + \frac{w_{11}^2(\theta)}{u_0(\theta)}\sin^2\tau - w_{11}(\theta)\frac{d}{\gamma}\right)\sin\Phi\,\frac{1-\zeta_2^2}{1+\zeta_2^2}\right\} + \dots \quad (5.16c)$$

The expansions for the density and normal velocity are

$$\rho = \rho_0(\theta) + \sigma\rho_0(\theta)\left\{\frac{d}{\gamma}\frac{1-\zeta_2^2}{1+\zeta_2^2} + \frac{1}{\gamma}\frac{p_{11}(\theta)}{p_0(\theta)}\cos\Phi - \frac{v_0'(\theta)\,d}{4\gamma w_{11}^2(\theta)\sec\tau}\left[2\frac{w_{11}(\theta)\,u_{11}(\theta)}{u_0(\theta)}\right.\right.$$
$$\left.\left. + 2\frac{w_{11}^2(\theta)}{u_0(\theta)}\sin^2\tau - \frac{w_{11}(\theta)\,d}{\gamma}\right]\left[\Theta^{\sigma 4w_{11}^2(\theta)\sec^2\tau/(v_0'(\theta))^2} - 1\right]\left[\frac{4\zeta_2^2}{(1+\zeta_2^2)^2}\frac{1-\zeta_2^2}{1+\zeta_2^2}\right]\right\}$$
$$+ \sigma^2\rho_0(\theta)\left\{\frac{1}{\gamma}\frac{p_{22}(\theta)}{p_0(\theta)}\cos 2\Phi + \frac{1}{2}\left(\frac{d}{\gamma}\frac{1-\zeta_2^2}{1+\zeta_2^2} + \frac{1}{\gamma}\frac{p_{11}(\theta)}{p_0(\theta)}\cos\Phi\right)^2 - \frac{1}{2\gamma}\left(\frac{p_{11}(\theta)}{p_0(\theta)}\right)^2\cos^2\Phi\right.$$
$$+ 2\left[\frac{\rho_{22}(\theta)}{\rho_0(\theta)} - \frac{1}{\gamma}\frac{p_{22}(\theta)}{p_0(\theta)} + \frac{1}{4\gamma}\left(\frac{p_{11}(\theta)}{p_0(\theta)}\right)^2 - \frac{1}{4}\left(\frac{\rho_{11}(\theta)}{\rho_0(\theta)}\right)^2\right.$$
$$\left. - \frac{1}{2\gamma}\frac{w_{11}(\theta)\,d}{v_0'(\theta)}\sec\tau\log\theta\right]\left(\frac{1-\zeta_2^2}{1+\zeta_2^2}\right)^2 - \frac{\rho_{22}(\theta)}{\rho_0(\theta)}$$
$$+ \frac{1}{\gamma}\frac{p_{22}(\theta)}{p_0(\theta)} + \frac{\rho_{20}(\theta)}{\rho_0(\theta)} + \frac{w_{11}(\theta)\,d}{\gamma v_0'(\theta)}\sec\tau\log\theta$$
$$- \frac{1}{2\gamma}\frac{d}{w_{11}(\theta)}\left[\frac{4p_{22}(\theta)}{\rho_0(\theta)\,u_0(\theta)} + 2\frac{w_{11}^2(\theta)}{u_0(\theta)}\sin^2\tau - \frac{w_{11}(\theta)}{\gamma}\frac{p_{11}(\theta)}{p_0(\theta)} + \frac{3}{2}\frac{w_{11}^2(\theta)}{u_0(\theta)}\right]$$
$$\left.\times\left[\frac{4\zeta_2^2}{(1+\zeta_2^2)^2}\right]\left[\log(1+\cos\Phi)^2 - \log\frac{4\zeta_2^2}{(1+\zeta_2^2)^2}\right]\right\} + \dots \quad (5.16d)$$

$$v = v_0(\theta) + \sigma\theta\left\{2[u_{11}(0) + w_{11}(0)\sin^2\tau]\frac{1-\zeta_2^2}{1+\zeta_2^2}\sec\tau\right.$$
$$+ w_{11}(0)\cos\Phi\,(2\sin^2\tau - 1)\sec\tau + \frac{v_0'(0)}{2w_{11}^2(0)}(u_{11}(0) + w_{11}(0)\sin^2\tau)\left[2\frac{w_{11}(0)\,u_{11}(0)}{u_0(0)}\right.$$
$$\left. + 2\frac{w_{11}^2(0)}{u_0(0)}\sin^2\tau - \frac{w_{11}(0)\,d}{\gamma}\right]\left[1 - \Theta^{4\sigma^2 w_{11}^2(0)\sec^2\tau/(v_0'(0))^2}\right]\frac{4\zeta_2^2}{(1+\zeta_2^2)^2}\frac{1-\zeta_2^2}{1+\zeta_2^2}$$
$$\left. - v_{11}'(0)\cos\Phi + \frac{1}{\theta}v_{11}(\theta)\cos\Phi\right\} + \sigma^2\theta\left\{\sec\tau\left[\left(\frac{4p_{22}(0)}{\rho_0(0)\,u_0(0)} + \frac{w_{11}^2(0)}{u_0(0)}\sin^2\tau\right.\right.\right.$$

$$
\begin{aligned}
&-\frac{w_{11}(0)}{\gamma}\frac{p_{11}(0)}{p_0(0)}-\frac{w_{11}^2(0)}{u_0(0)}\Big)\left(\frac{\sin^2\tau}{2}-1\right)-\frac{w_{11}(0)}{2\gamma}\frac{p_{11}(0)}{p_0(0)}\Big]\cos 2\Phi \\
&-2\sec\tau\left[2u_{22}(0)+\frac{w_{11}(0)}{v_0'(0)}(u_{11}(0)+w_{11}(0)\sin^2\tau)\sec\tau\log\theta+\sin^2\tau\left(\frac{2p_{22}(0)}{\rho_0(0)\,u_0(0)}\right.\right. \\
&\left.\left.-\frac{w_{11}^2(0)}{2u_0(0)}\sin^2\tau-\frac{w_{11}(0)}{2\gamma}\frac{p_{11}(0)}{p_0(0)}-\frac{w_{11}^2(0)}{2u_0(0)}-\frac{w_{11}(0)\,u_{11}(0)}{u_0(0)}+\frac{w_{11}(0)\,d}{\gamma}\right)\right] \\
&\times\left(\frac{1-\zeta_2^2}{1+\zeta_2^2}\right)^2+\sec\tau\left[2u_{22}(0)-2u_{20}(0)+\sin^2\tau\left(\frac{2p_{22}(0)}{\rho_0(0)\,u_0(0)}\right.\right. \\
&\left.+\frac{w_{11}^2(0)}{2u_0(0)}\sin^2\tau-\frac{w_{11}(0)}{2\gamma}\frac{p_{11}(0)}{p_0(0)}-\frac{w_{11}(0)}{2\gamma}\frac{p_{11}(0)}{p_0(0)}-\frac{w_{11}^2(0)}{2u_0(0)}\right) \\
&\left.+\frac{2w_{11}(0)}{v_0'(0)}(u_{11}(0)+w_{11}(0)\sin^2\tau)\sec\tau\log\theta\right]+\sec\tau\,(2\sin^2\tau-1)\left[\frac{w_{11}(0)\,u_{11}(0)}{u_0(0)}\right. \\
&\left.+\frac{w_{11}^2(0)}{u_0(0)}\sin^2\tau-\frac{w_{11}(0)\,d}{\gamma}\right]\cos\Phi\,\frac{1-\zeta_2^2}{1+\zeta_2^2}-\frac{\sec\tau}{w_{11}(0)}(u_{11}(0) \\
&+w_{11}(0)\sin^2\tau)\left[\frac{4p_{22}(0)}{\rho_0(0)\,u_0(0)}-2\frac{w_{11}^2(0)}{u_0(0)}\sin^2\tau-\frac{w_{11}(0)}{\gamma}\frac{p_{11}(0)}{p_0(0)}-\frac{3}{2}\frac{w_{11}^2(0)}{u_0(0)}\right] \\
&\times\frac{4\zeta_2^2}{(1+\zeta_2^2)^2}\left[\log(1+\cos\Phi)^2-\log\frac{4\zeta_2^2}{(1+\zeta_2^2)^2}\right]-\sec\tau\left[\frac{2w_{11}(0)\,u_{11}(0)}{u_0(0)}\right. \\
&\left.+\frac{2w_{11}^2(0)}{u_0(0)}\sin^2\tau-\frac{w_{11}(0)\,d}{\gamma}\right]\frac{4\zeta_2^2}{(1-\zeta_2^2)^2}-v_{20}'(0)-v_{22}'(0)\cos 2\Phi \\
&\left.+(1/\theta)\,(v_{20}(\theta)+v_{22}(\theta)\cos 2\Phi)\right\}+\ldots. \qquad (5.16e)
\end{aligned}
$$

The previous expansion for the pressure, equation (3.1d), is uniformly valid to second order in σ. The uniformly valid expansion for the entropy can be found by substituting (5.16d) and (3.1d) into equation (3.12). The result is

$$
\begin{aligned}
\frac{\Delta s}{c_v}&=-\sigma\left\{d\frac{1-\zeta_2^2}{1+\zeta_2^2}-\frac{v_0'(\theta)\,d}{4w_{11}^2(\theta)\sec\tau}\left[2\frac{w_{11}(\theta)\,u_{11}(\theta)}{u_0(\theta)}\right.\right. \\
&\left.\left.+2\frac{w_{11}^2(\theta)}{u_0(\theta)}\sin^2\tau-\frac{w_{11}(\theta)\,d}{\gamma}\right]\left[\Theta^{\sigma 4w_{11}^2(\theta)\sec^2\tau/(v_0'(\theta))^2}-1\right]\left[\frac{4\zeta_2^2}{(1+\zeta_2^2)^2}\frac{1-\zeta_2^2}{1+\zeta_2^2}\right]\right\} \\
&-\sigma^2\left\{2\left[\gamma\frac{\rho_{22}(\theta)}{\rho_0(\theta)}-\frac{p_{22}(\theta)}{p_0(\theta)}+\frac{1}{4}\left(\frac{p_{11}(\theta)}{p_0(\theta)}\right)^2-\frac{\gamma}{4}\left(\frac{\rho_{11}(\theta)}{\rho_0(\theta)}\right)^2-\frac{1}{2}\frac{w_{11}(\theta)\,d}{v_0'(\theta)}\sec\tau\log\theta\right]\right. \\
&\times\left(\frac{1-\zeta_2^2}{1+\zeta_2^2}\right)^2-\gamma\frac{\rho_{22}(\theta)}{\rho_0(\theta)}+\frac{p_{22}(\theta)}{p_0(\theta)}+\gamma\frac{\rho_{20}(\theta)}{\rho_0(\theta)}-\frac{p_{20}(\theta)}{p_0(\theta)} \\
&+\frac{w_{11}(\theta)\,d}{v_0'(\theta)}\sec\tau\log\theta-\frac{1}{2}\frac{d}{w_{11}(\theta)}\left[\frac{4p_{22}(\theta)}{\rho_0(\theta)\,u_0(\theta)}+2\frac{w_{11}^2(\theta)}{u_0(\theta)}\sin^2\tau\right. \\
&\left.\left.-\frac{w_{11}(\theta)}{\gamma}\frac{p_{11}(\theta)}{p_0(\theta)}+\frac{3}{2}\frac{w_{11}^2(\theta)}{u_0(\theta)}\right]\left[\frac{4\zeta_2^2}{(1+\zeta_2^2)^2}\right]\left[\log(1+\cos\Phi)^2-\log\frac{4\zeta_2^2}{(1+\zeta_2^2)^2}\right]\right\}+\ldots. \\
&\qquad (5.16f)
\end{aligned}
$$

5.5. *Discussion*

To obtain a picture of the flow it is sufficient to examine the first- and second-order expansions for the entropy, since lines of constant entropy are projections of the streamlines onto any surface r = const.

From (5.13) it can be seen that in the outer region Δs is constant on lines of constant ϕ as in figure 1. Near the surface, streamlines bend around the body toward the top ray. ζ_1 varies from zero to infinity, the cone surface and the streamline in the plane of symmetry on the windward side of the cone corresponding to $\zeta_1 = 0$. From this fact it is clear that the body is wetted by the streamline carrying the maximum entropy. The streamline in the leeward plane of symmetry carries the minimum entropy and is described by $\zeta_1 = \infty$. These two streamlines meet—as do all other streamlines—at $\theta = 0$, $\phi = 0$ (see figure 3).

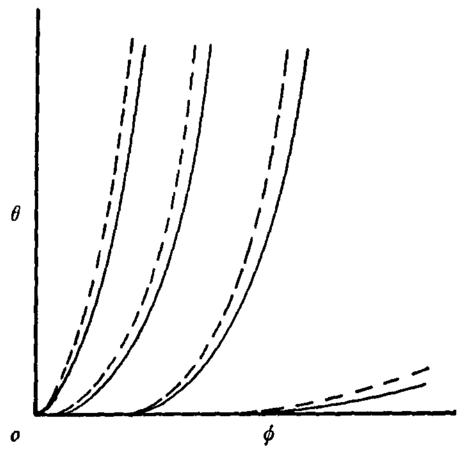

FIGURE 3. Projection of streamlines in the neighbourhood of the vortical singularity on r = const. surface (schematic). ——, First-order streamlines; ---, second-order streamlines.

Thus it is clear that the first-order solution represents Ferri's vortical layer together with the vortical singularity at $\theta = 0$, $\phi = 0$. The analytical results of Cheng, Bulakh, Woods, Sapunkov and Melnik are in agreement with these results.

The second-order perturbation is a correction to the vortical layer solution. The surface streamline and the streamlines in the windward and leeward planes of symmetry are unaltered. As shown in figure 1, the other streamlines depart slightly from lines of constant ζ_2. Near the vortical singularity, however, $\Delta s/cv$ is a function only of ζ_2. Thus, in this neighbourhood we can approximate the streamlines by

$$\theta^{\sigma f_1(0)+\sigma^2 f_2(0)}/\phi^2 = \text{const.} \tag{5.17}$$

The functions f_1 amd f_2 are those in the exponent of Θ in (5.16*b*). Clearly, the second-order correction to the streamline shape in the neighbourhood of the vortical singularity—i.e. $\sigma^2 f_2(0)$—alters the streamline shape slightly but does not change the analytic nature of the curves (see figure 3).

The second-order perturbations agree in form with those of Sapunkov everywhere except in the neighbourhood of the vortical singularity. In that region Sapunkov's result for the streamline shape is

$$\theta^B/\phi^2 = \text{const.}, \quad \text{where} \quad B = \sigma g_1 + \sigma^2 g_2(\theta^{\sigma g_1}/\phi^2) \tag{5.18}$$

and g_1 is a constant. It is clear this does not agree with (5.17) since θ/ϕ^2 takes on all values from zero to infinity near the vortical singularity. To compare these two results, we consider generalizing them to nth order. We would obtain for the function in (5.17) an nth-order polynomial in σ in the exponent of θ, whereas for the function in (5.18) we would obtain a very complicated expression with an escalating exponent. If we substitute this latter function into the equations of motion we arrive at a result that is not consistent with any sensible local expansion. It is on this basis that we prefer (5.17).

Ferri (1950) has conjectured that at high angles of attack the vortical singularity leaves the surface of the cone. Although our result is valid only for small angle of attack we may reasonably ask if it supports Ferri's conjecture. We might suspect that the additional non-uniformity that appears in (5.14) is connected with this phenomenon. If we assume that this is true, it follows that the vortical singularity will be located at

$$\Phi = 0, \quad \Theta = k\sigma,$$

where k is a constant. Then the first-order terms will be of the form:

$$\frac{1-(\Theta-k\sigma)^{f_1+\sigma f_2}(1+\cos\Phi)/(1-\cos\Phi)}{1+(\Theta-k\sigma)^{f_1+\sigma f_2}(1+\cos\Phi)/(1-\cos\Phi)}.$$

We now expand this expression for small σ and try to identify the result with terms in (5.14). The result of the expansion is

$$\frac{1-\Theta^{f_1}(1+\cos\Phi)/(1-\cos\Phi)}{1+\Theta^{f_1}(1+\cos\Phi)/(1-\cos\Phi)} - 2\sigma\frac{\Theta^{f_1}(1+\cos\Phi)/(1-\cos\Phi)}{(1+\Theta^{f_1}(1+\cos\Phi)/(1-\cos\Phi))^2} \times (f_2\log\Theta - f_1 k\Theta^{-1}) + \dots.$$

Since no term proportional to Θ^{-1} appears in (5.14), we must conclude that to second order in angle of attack the singularity remains on the surface. The appearance of such a term in higher-order perturbations would be an indication that the singularity leaves the surface.

Numerical values for the first-order outer quantities can be obtained by applying the transformations of Roberts & Riley (1954) to the results of Kopal (1947*b*). Since Kopal (1949) ignored the logarithmic singularities in the computation of the second-order quantities, the results of Roberts & Riley probably approximate closely the correct second-order quantities with the singularities removed. For example, the second-order density perturbations one would obtain from applying Roberts & Riley's formulas to Kopal's result probably closely approximate

$$\rho_{2m} + (-1)^{\frac{1}{2}m}(w_{11}d/2\gamma J v_0')\log\theta.$$

The same is true for the radial velocity perturbation.

Our solution displays the basic features of Ferri's vortical layer, the only disagreement being with Ferri's estimate of the thickness of the layer. The present result and also the results of Cheng, Bulakh, Woods, and Melnik indicate that the layer is thinner than any power of σ. The solution carried out here to second order in σ could be extended to any order, though a certain amount of ingenuity would be required to treat the vortical singularity itself to higher order.

The results of Bulakh (1962*a*) are in basic agreement with the present first-order solution. His result for the radial velocity is identical with (5.10) if one changes $u_1^\times$ to $u_1(\theta)$ in his equation (5.4). The present results correct the non-uniformity in normal velocity gradient that he overlooked. The solution of Woods (1962) contains some of the features of the present first-order solution, but no explicit results for the flow variables throughout the flow field are given.

The present first-order solution can be compared directly with the results of Cheng and Sapunkov (1963*a*, *b*) by taking the Newtonian limit of the zero-incidence and first-order quantities. This is done by writing u_0, v_0/ϵ, p_0, and $\epsilon\rho_0$ as functions of ϕ and θ/ϵ, and expanding them in ascending powers of

$$\epsilon = (\gamma-1)/(\gamma+1).$$

The process has been carried far enough to check first-order results, i.e. coefficients of σ and ϵ. The results obtained agree with Cheng's corrected results in the outer region and with those of Sapunkov. The Newtonian limit of the inner expansion is obtained in the same way, except that the independent variables are $(\theta/\epsilon)^{\sigma\epsilon}$ and ϕ. With the exception of the normal component of velocity, these results agree with those of Cheng. The present results confirm the presence of the non-uniformity in normal velocity that was discovered by Sapunkov (1963*b*), but show that he made an algebraic error in removing that non-uniformity. One may obtain the correct result by replacing the coefficient of $\theta\zeta$ in the first of his equations (7.2) by $-2\sin^2\tau\sin\omega/\cos\tau$.

The second-order terms are corrections to the basic vortical layer solution. The streamline shapes are altered in the entire flow field. In particular, in the neighbourhood of the vortical singularity they are given by (5.17). The unpublished results of Melnik are in agreement with (5.17). Kopal's tabulated results can be corrected to second order everywhere by using equations (5.16).

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